PRINCIPAL IDEALS IN SUBALGEBRAS OF GROUPOID C*-ALGEBRAS

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Abstract. The study of different types of ideals in non self-adjoint operator algebras has been a topic of recent research. This study focuses on principal ideals in subalgebras of groupoid C*-algebras. An ideal is said to be principal if it is generated by a single element of the algebra. We look at subalgebras of r-discrete principal groupoid C*-algebras and prove that these algebras are principal ideal algebras. Regular canonical subalgebras of almost finite C*-algebras have digraph algebras as their building blocks. The spectrum of almost finite C*-algebras has the structure of an r-discrete principal groupoid and this helps in the coordinization of these algebras. Regular canonical subalgebras of almost finite C*-algebras have representations in terms of open subsets of the spectrum for the enveloping C*-algebra. We conclude that regular canonical subalgebras are principal ideal algebras.

Introduction

Non self-adjoint limit algebras are direct limits of subalgebras of finite dimensional C*-algebras. The refinement embedding algebra, standard embedding algebra and the alternating embedding algebra are examples of non self-adjoint limit algebras and these examples are explained in detail in this paper. Some important special classes of non self-adjoint limit algebras are :

- TUHF(Triangular Uniformly Hyperfinite) algebras.
- TAF(Triangular approximately finite dimensional) algebras.
- Regular canonical subalgebras.

TUHF algebras form a subclass of TAF algebras and TAF algebras form a subclass of regular canonical subalgebras. The upper triangular complex matrix algebra T_n is a basic building block for TUHF algebras whereas direct sums of upper triangular matrix algebras are the basic building blocks for the TAF algebras. Digraph algebras are the building blocks for regular canonical subalgebras of AF C*-algebras. The refinement embedding algebra, standard embedding algebra and the alternating embedding algebra are examples of TUHF limit algebras. The study of different types of ideals in non self-adjoint operator algebras has been a topic of recent research. We will study one of the basic types of ideals: a principal ideal in some non self-adjoint limit algebras. An ideal is said to be principal if it is generated by a single element of the algebra. In this study we will first analyze the structure of ideals in digraph algebras and prove that digraph algebras are principal ideal

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algebras. Since regular canonical subalgebras of AF C*-algebras are infinite dimensional analogues of digraph algebras, it is natural to expect these algebras to be principal ideal algebras. But it is observed that the proof does not follow naturally and we have to resort to the spectrum of these algebras in the proof. The spectrum of AF C*-algebras has the structure of an r-discrete principal groupoid and it is this groupoid and substructures of the groupoid which help in the coordinization of these algebras. Any closed subalgebra of an AF C*-algebra which contains the diagonal has a functional representation in terms of an open subset of the spectrum of the enveloping C*-algebra and we use this representation to prove that regular canonical subalgebras of AF C*-algebras are principal ideal algebras.

1. Preliminaries

In this section we define AF C*-algebras, UHF C*-algebras, TAF algebras, TUHF algebras and regular canonical subalgebras of AF C*-algebras. TAF and TUHF algebras are non self-adjoint versions of AF C*-algebras and UHF C*-algebras respectively. We will illustrate the definitions with some examples. Also there are two equivalent ways of defining these algebras, one as an inductive limit and other as the closure of an increasing union of finite dimensional algebras and these are explained below.

1.1. **Direct limit of C*-algebras.**

Definition 1.1.1. A C*-algebra is a norm closed self-adjoint subalgebra of the operator algebra B(H), for some Hilbert space H.

Every finite dimensional C*-algebra is *-isomorphic to the direct sum of full matrix algebras (A proof of this is indicated in [2], page 74). Next let $B_1 \overset{\varphi_1}{\to} B_2 \overset{\varphi_2}{\to} B_3 \overset{\varphi_3}{\to} B_4 \overset{\varphi_4}{\to} \cdots B_n \overset{\varphi_n}{\to} B_{n+1} \cdots$ denote an injective direct system of C*-algebras $B_n, n = 1, 2, \cdots$ with star injections $\varphi_n : B_n \to B_{n+1}$. Then the product $\prod_{n=1}^{\infty} B_n$ is a *-algebra with pointwise defined operations. Let $B_\infty^0 = \{b : b = (b_n) \in \prod_{n=1}^{\infty} B_n, \varphi_n(b_n) = b_{n+1},$ for all large $n\}$. Then B_∞^0 is a *-subalgebra of $\prod_{n=1}^{\infty} B_n$. Any *-homomorphism $\varphi_n : B_n \to B_{n+1}$ is necessarily norm-decreasing (this is a standard result from C*-algebras). Consequently $\|b_{n+1}\| \leq \|b_n\|$ for large n and so the sequence $(\|b_n\|)$ is eventually decreasing and also bounded below. Thus the sequence $(\|b_n\|)$ converges. Let $p(b) = \lim_{n\to\infty} \|b_n\|$. Then it is clear that $p : B_\infty^0 \to R^+$, $b \mapsto p(b)$, is a C*-seminorm on B_∞^0. We denote the enveloping C*-algebra of (B_∞^0, p) by B, and call it the *direct limit* of the sequence $(B_n, \varphi_n)_{n=1}^{\infty}$. Also we denote B by $\lim\limits_{n} B_n$.

Next we define AF C*-algebra and UHF C*-algebra.

Definition 1.1.2. A C*-algebra B is approximately finite dimensional (AF) if it is the closure of an increasing union of finite-dimensional C*-subalgebras B_n.

Remark 1.1.3. Equivalently every AF C*-algebra is *-isomorphic to a direct limit algebra, $\lim\limits_{i}(B_{n_i}, \phi_i)$, associated with standard unital injective maps $\phi_i : B_{n_i} \to B_{n_{i+1}}$, where each B_{n_i} is a direct sum of full matrix algebras.

A simple finite dimensional C*-algebra is *-isomorphic to a full matrix algebra.

Definition 1.1.4. A C*-algebra B is uniformly hyperfinite (UHF) if it is the closure of an increasing union of simple finite-dimensional C*-subalgebras B_n.

Remark 1.1.5. Equivalently, every UHF algebra is *-isomorphic to a limit algebra, $\lim_i(M_{n_i}, \phi_i)$, associated with standard unital injective maps $\phi_i : M_{n_i} \to M_{n_{i+1}}$.

1.2. **Matrix unit system for limit algebras.** Since $M_n \cong B(\mathbb{C}^n)$, where $\mathbb{C}^n$ denotes the n-dimensional complex field, we will make frequent use of the fact that corresponding to any orthonormal basis $e_1, e_2, \cdots, e_n$, M_n has a basis consisting of matrix units $E_{ij} = e_i e_j^*$ (e_j^* denotes the conjugate transpose of e_j) for $1 \leq i, j \leq n$. We can also define $e_i e_j^*$ as an operator: $e_i e_j^*(x) = < x, e_j > e_i$. Consequently if $B \cong M_{n_1} \oplus M_{n_2} \oplus M_{n_3} \oplus \cdots M_{n_k}$, then B has a matrix unit system(m.u.s), say $\{E_{ij}^s : 1 \leq s \leq k, 1 \leq i, j \leq n_s\}$. Evidently such a m.u.s is not unique because if $\{E_{ij}\}$ is a m.u.s then $\{e^{i\theta(j-i)}E_{ij}\}$ is another m.u.s where $e^{i\theta}$ denotes a complex number of modulus 1. Also conjugating the m.u.s by any unitary will yield other matrix unit systems. But any two m.u.s of a finite dimensional C*-algebra are inner conjugate and hence the choice of m.u.s for a chain of finite dimensional C*-algebras is irrelevant although it does matter how the m.u.s fits with the embeddings. The m.u.s for a chain $\{B_k\}$ is a system $\{e_{ij}^k\}_{ijk}$ where for each k the system $\{e_{ij}^k\}_{ij}$ is a m.u.s for B_k and where each e_{ij}^k is a sum of the elements of $\{e_{ij}^{k+1}\}_{ij}$. Suppose $B = \overline{\cup_{n=1}^{\infty} A_n}$ is an AF algebra and let D_n be a maximal abelian self-adjoint subalgebra (masa) for each n such that $D_n \subseteq D_{n+1}$, for all n. Let $D = \overline{\cup_{i=1}^{\infty} D_n}$. Then D is a masa of B and $D_n = D \cap B_n$ is such that $D_n \subseteq D_{n+1}$. The existence of such a masa is guaranteed in [6]. Let $B_n = \oplus_{m=1}^{l(n)} M_{k(n,m)}$, where M_k denotes a $k \times k$ matrix. Then for each n and m, a m.u.s $\{e_{ij}^{nm}\}$ can always be chosen for $M_{k(n,m)}$ so that if $\phi_n : B_n \to B_{n+1}$ denotes the embedding from B_n to B_{n+1} then $\phi_n(e_{ij}^{(nm)})$ is a sum of matrix units of B_{n+1}. Also the m.u.s can be chosen such that each D_n is generated by the diagonal matrix units. Consequently D is the closed linear span of $\{e_{ii}^{(nm)} : 1 \leq n, 1 \leq m \leq l(n), 1 \leq i \leq k(n,m)\}$. (The reader is referred to [16] for details.)

Remark 1.2.1. All subalgebras of AF algebras in this paper are norm closed.

Let D be the abelian C*-algebra generated by all of the diagonal matrix units $e_{ii}^{(nm)}$ associated with the m.u.s $\{e_{ii}^{(nm)}\}$, as above. Then $D = \overline{\cup_{i=1}^{\infty} D_n}$, where D_n is the masa in B_n spanned by the diagonal matrix units. Then D is a masa in B (for a proof of this, refer to [16]). We call D a *regular canonical masa* associated with the m.u.s. $\{e_{ij}^{(nm)}\}$.

At this point we will define TAF algebras and TUHF algebras and regular canonical subalgebras. Let A denote a regular canonical subalgebra of an AF C*-algebra B. It is important to note that the embeddings $\phi_i : A_{n_i} \to A_{n_{i+1}}$ are *-extendible and maps the normaliser of D_k into the normaliser D_{k+1}. Such embeddings are called *regular embeddings* in literature.

Definition 1.2.2. A regular canonical subalgebra A of an AF C*-algebra B is a closed subalgebra of B such that $D \subseteq A \subseteq B$; where D is a regular canonical masa associated with a m.u.s for B.

Definition 1.2.3. If $B = \overline{\cup_{i=1}^{\infty} B_n}$ is an AF algebra with masa D , then a subalgebra A of B is said to be TAF with diagonal D if $D = A \cap A^*$.

Definition 1.2.4. If $B = \overline{\cup_{i=1}^{\infty} B_n}$ is a UHF algebra with masa D , then a subalgebra A of B is said to be TUHF with diagonal D if $D = A \cap A^*$.

Definition 1.2.5. A triangular subalgebra A of an AF C*-algebra B is a closed subalgebra of B such that $A \cap A^*$ is a masa.

Definition 1.2.6. A TAF subalgebra A of an AF C*-algebra B is said to be maximal triangular if A is the only triangular subalgebra of B containing A.

Remark 1.2.7. In the above definition, if the sequence $\{B_n\}$ can be chosen such that $A \cap B_n$ is maximal triangular in B_n for each n, then A is called strongly maximal triangular.

The refinement embedding algebra, standard embedding algebra and the alternating embedding algebra are examples of strongly maximal triangular algebras. It is evident that a strongly maximal TAF algebra is a maximal TAF algebra. But the converse is not true and an example is given in [14].

Remark 1.2.8. If A is a TUHF algebra then it may be possible to write $A = \overline{\cup_{i=1}^{\infty} A_n}$ where each A_n is not a factor. This motivates the following definition.

Definition 1.2.9. A strongly maximal triangular subalgebra A of a UHF algebra B is said to be strongly maximal in factors if a sequence $\{B_n\}$ can be chosen such that $B_n \cong M_{n_k}$ for each n, $B = \overline{\cup_{i=1}^{\infty} B_n}$ and $A \cap B_n$ is maximal triangular in B_n for each n.

Again it is not true in general that a strongly maximal TUHF algebra is strongly maximal in factors. An example is given in [14].

1.3. **Examples of limit algebras.** We will study the refinement embedding algebra, standard embedding algebra, the alternating embedding algebra and digraph algebras in detail.

1. Let (n_k) denote a sequence of positive integers such that n_k divides n_{k+1}, for each $k = 1, 2, \cdots$ Consider the unital injective maps $\rho_k : M_{n_k} \rightarrow M_{n_{k+1}}$ given by $\rho_k(a_{ij}) = (a_{ij} I_{r_k})$ such that $(a_{ij} I_{r_k})$ is the partitioned matrix in $M_{n_{k+1}}$, with I_{r_k} the identity matrix in M_{r_k}, where $r_k = \frac{n_{k+1}}{n_k}$. In this case $\rho_k(e_{ij}^k) = \sum_{t=1}^{q_k} e^{k+1}_{(i-1)q_k+t,(j-1)q_k+t}$. Here is an example of a refinement embedding :

$$\rho_k(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \left[\begin{array}{cccccc|cccccc} \mathrm{a} & 0 & . & . & 0 & 0 & \mathrm{b} & 0 & . & . & 0 & 0 \\ 0 & \mathrm{a} & . & . & 0 & 0 & 0 & \mathrm{b} & . & . & 0 & 0 \\ . & . & . & . & 0 & 0 & . & . & . & . & 0 & 0 \\ . & . & . & . & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{a} & 0 & 0 & 0 & 0 & 0 & \mathrm{b} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{a} & 0 & 0 & 0 & 0 & 0 & \mathrm{b} \\ \hline \mathrm{c} & 0 & . & . & 0 & 0 & \mathrm{d} & 0 & . & . & 0 & 0 \\ 0 & \mathrm{c} & . & . & 0 & 0 & 0 & \mathrm{d} & . & . & 0 & 0 \\ . & . & . & . & 0 & 0 & . & . & . & . & 0 & 0 \\ . & . & . & . & 0 & 0 & . & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{c} & 0 & 0 & 0 & 0 & 0 & \mathrm{d} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{c} & 0 & 0 & 0 & 0 & 0 & \mathrm{d} \end{array}\right]$$

where the righthand side is a 2 by 2 matrix, with each entry amplified by order r using the identity matrix I_r. Then the limit algebra $\lim_k(M_{n_k}, \rho_k)$ associated with these unital injective maps $\rho_k : M_{n_k} \to M_{n_{k+1}}$ is an example of a UHF C*-algebra (also called Glimm Algebra in literature). Then if $T_n \subseteq M_n$ denotes the algebra of upper triangular matrices relative to the standard m.u.s it follows that $\rho_k(T_{n_k}) \subseteq T_{n_{k+1}}$. The canonical subalgebra $\lim_k(T_{n_k}, \rho_k)$ of the UHF C*-algebra is called a refinement limit algebra.

2. Let (n_k) denote a sequence of positive integers such that n_k divides n_{k+1}, for each $k = 1, 2, \cdots$ Consider the unital injective maps $\sigma_k : M_{n_k} \to M_{n_{k+1}}$ given by $\sigma_k(e^k_{ij}) = \sum_{t=0}^{q_k-1} e^{k+1}_{i+tq_k, j+tq_k}$. The unital injective maps $\sigma_k : M_{n_k} \to M_{n_{k+1}}$ are called standard embeddings. Here is an example of a standard embedding :
$$\sigma_k(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = \begin{bmatrix} a & b & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & a & b & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & c & d & 0 & 0 & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b & . & . & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & d & . & . & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & a & b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & c & d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & 0 & 0 & . & . & 0 & 0 & c & d \end{bmatrix}$$
Then if $T_n \subseteq M_n$ denotes the algebra of upper triangular matrices relative to the standard m.u.s it follows that $\sigma_k(T_{n_k}) \subseteq T_{n_{k+1}}$. The canonical subalgebra $\lim_k(T_{n_k}, \rho_k)$ is called a standard limit algebra.

3. Alternation limit algebras are limit algebras in which the refinement embeddings and the standard embeddings are used alternatively. Let (r_k) and (s_k) denote sequences of positive integers; $k = 1, 2, \cdots$. Let $\sigma_k : M_{s_k} \to M_{s_{k+1}}$ denote the standard embeddings and $\rho_k : M_{r_k} \to M_{r_{k+1}}$ denote the refinement embeddings. Then the limit of the direct system $C \xrightarrow{\rho} T_{r_1} \xrightarrow{\sigma} T_{r_1 s_1} \xrightarrow{\rho} T_{r_1 s_1 r_2} \xrightarrow{\sigma} \cdots$ is called an alternation limit algebra.

4. A digraph algebra is a subalgebra of the full complex matrix algebra M_n which contains a maximal abelian subalgebra of M_n. Let D_n denote the standard diagonal algebra associated with the standard m.u.s . The digraph algebra is unitarily equivalent to an algebra containing D_n. Thus an example of a digraph algebra is
$$\begin{bmatrix} * & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & 0 \\ 0 & * & * & * & 0 & * & 0 \\ 0 & * & * & * & 0 & * & 0 \\ 0 & * & 0 & 0 & * & * & 0 \\ 0 & * & * & * & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}.$$
Digraph algebras are basic building blocks for regular canonical subalgebras. Let A denote a digraph algebra such that $A \subseteq M_n$. Then we have $D_n \subseteq A \subseteq M_n$, where D_n denotes the algebra of diagonal matrices. Let

Lat A denote the lattice of invariant projections of A. Then Lat $A = \{p \in M_n : p = p^* = p^2$ and $ap = pap, \forall a \in A\}$. Let $p \in$ Lat A. Then p is also invariant for D_n. But D_n is a self-adjoint algebra and so p is invariant for D_n implies p lies in the *commutant* of D_n, where *commutant* is the set of all elements of M_n that commute with all elements of D_n and is denoted by D_n^c. But D_n is a masa and so $D_n^c = D_n$. So $p \in D_n$. Since p is an arbitrary element in Lat A, we can infer that Lat $A \subseteq D_n$ and consequently is a commutative lattice. Note that Lat A is a commutative lattice with $p \vee q = p + q - pq$ and $p \wedge q = pq$ for all elements $p, q \in$ Lat A. Thus the lattice of invariant projections for a digraph algebra is a commutative subspace lattice(CSL). So the digraph algebras are just the finite dimensional CSL algebras.

Remark 1.3.1. If $\frac{n_{k+1}}{n_k} = 2$ for $k = 1, 2, \cdots$ in examples 1 and 2 above, then the standard limit algebra and the refinement limit algebra are called 2^∞ TUHF algebras.

2. Isometric isomorphism of limit algebras

The UHF algebras corresponding to the refinement and standard embeddings are isometrically isomorphic. This is proved in [6] by J.Glimm. In fact Glimm proved that the isomorphism class was independent of the nature of the embeddings and depended only on the dimensions of the finite dimensional factors. But this is not true in the case of TUHF algebras. In fact the standard limit algebra and the refinement limit algebra are not isometrically isomorphic. Let S and R denote the 2^∞ TUHF algebras via the standard embedding and refinement embedding respectively and let Lat S and Lat R denote the lattice of invariant projections of S and R respectively. Then Lat $S = \{0, 1\}$ and Lat $R =$ L where $L = \{\sum_{i=1}^{j} e_{ii}^n : 1 \leq j \leq 2^n, 1 \leq n \leq \infty\}$ $\cup\{0\}$, a nest. Thus the limit algebras are not isomorphic, since Lat S and Lat R are not isomorphic. Also it is interesting to observe that the refinement embedding maps Lat T_{n_k} into Lat $T_{n_{k+1}}$ whereas the standard embedding does not. The classification of refinement limit algebra, standard limit algebra and the alternating limit algebra up to isometric isomorphism has been done in [7]. For all classifications the authors used the spectrum, also called the topologized fundamental relation. The results in this paper also make use of the spectrum. The spectrum of these algebras has been studied in detail in [16]. But for the sake of completeness we will describe the spectrum of these algebras.

3. Spectrum or Fundamental relation

3.1. **Introduction.** To understand the concept of spectrum for a limit algebra, we will start by defining a normalising partial isometry and we will see how normalising partial isometries act on the maximal ideal space of the canonical masa of a limit algebra to generate the spectrum.

3.2. **Spectrum of limit algebras.** Let B denote a limit algebra $\lim_i(B_i)$ arising from the direct system $B_1 \xrightarrow{\varphi_1} B_2 \xrightarrow{\varphi_2} B_3 \xrightarrow{\varphi_3} B_4 \xrightarrow{\varphi_4} \cdots B_n \xrightarrow{\varphi_n} B_{n+1} \cdots$ such that the *-extendible embeddings φ_i map the matrix units of B_i to sums of matrix units of B_{i+1}. Let D denote the regular canonical masa of B associated with a m.u.s. Then with $D_n = D \cap B_n$, $D_n \subseteq D_{n+1}$ and $D = \overline{\cup_{n=1}^{\infty} D_n}$.

Definition 3.2.1. An element p in B is a projection if $p^* = p = p^2$.

Definition 3.2.2. An element v in B is a partial isometry if v^*v is a projection.

Definition 3.2.3. The range (final) projection and the domain (initial) projection of a partial isometry v in B is defined as $r(v) = vv^*$ and $d(v) = v^*v$ respectively.

Definition 3.2.4. A map α is a partial homomorphism of the topological space X if the domain $d(\alpha)$ and the range $r(\alpha)$ are clopen subsets of X and α is a homeomorphism of $d(\alpha)$ onto $r(\alpha)$.

Definition 3.2.5. A partial isometry v in B_i is called a normalising partial isometry if $vD_iv^* \subseteq D_i$ and $v^*D_iv \subseteq D_i$ where D_i denotes the masa in B_i.

Remark 3.2.6. The normaliser $N_{D_i}(B_i)$ is the set of normalising partial isometries of D_i in B_i. For example if B_i is the upper triangular matrix algebra then the normaliser is the set of all upper triangular matrices with entries either 0 or of absolute value 1 such that each row or column has at most one non-zero entry.

Although the spectrum of limit algebras has been described in detail in literature, we will describe it and then illustrate it by working out the spectrum for some specific examples. Let B denote an AF algebra. Let $D \subseteq B$ be a canonical masa and let X denote the Gelfand spectrum. If $x \in X$, then there is a decreasing sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $C(X)$ with $\cap_{n=1}^{\infty} \hat{p_n} = \{x\}$; where $\hat{p_n}$ denotes the spectrum of p_n in X. In other words $\hat{p_n}$ is the image of p_n under the Gelfand map. Let $\{e_{ij}^{(n)}\}$ be a set of matrix units of B with respect to D. Then, p_n can be chosen as a diagonal matrix unit in B_n. Also note that once you have picked the projections p_n, you cannot be sure that $p_n \in B_n$, only that $p_n \in B_{k_n}$ for some k_n. This is just as good. If v is a matrix unit in B with $x \in \widehat{vv^*}$ then there is an $n \in N$ such that for $n \geq N$,$\{v^*p_nv\}_{n \geq N}$ forms a decreasing set of diagonal projections and the intersection $\cap_{n=1}^{\infty} v^*\hat{p}_nv$ is a singleton, say y. If $\hat{v}$ denotes the graph of v, we write $\sigma_v(x) = y \Leftrightarrow (x, y) \in \hat{v}$. In this way, v is viewed as a partial homeomorphism of X, with domain $r(v) = \widehat{vv^*}$ and range $d(v) = \widehat{v^*v}$. The orbit of x is denoted by $[x]$ with $[x] = \{\sigma_v(x) : v \text{ is a matrix unit of } B \text{ with } x \in \widehat{vv^*}\}$. Each of these orbits is countable. If $A \subset B$ is a TAF algebra with $A \cap A^* = D$; we define a partial order on each equivalence class in X. We call $x \leq y$ if $\sigma_v(x) = y$ for some matrix unit $v \in A$. This is the partial order. This is a total order on each equivalence class iff A is strongly maximal (This is discussed in Chapter 4). Let $R = \cup\{\hat{v} : v \text{ is a matrix unit of } B\}$. Then $R \subset X \times X$. R is topologized by letting the compact open sets $\hat{v}$ form a base for the topology. If $P = \cup\{\hat{v} : v \text{ is a matrix unit of } A\}$, then $P \subset R$ is called the fundamental relation or spectrum of A. More precisely, the sets $\hat{v}$ form a base for the topology: They turn out to be compact in this topology. To summarize; let E_{ij}^k denote the set of points (x,y) in $X \times X$ of the form $(\alpha(y), y)$ where α is

the partial homeomorphism of X induced by e_{ij}^k and y belongs to the domain of α. Then $P = R(A) = \cup\{E_{ij}^k : e_{ij}^k \in A_k, k = 1, 2, \cdots\}$ denotes the topological binary relation of A with the relative topology. The topological binary relation $R(A)$ is the spectrum of A.

3.3. **Examples of spectrum for certain limit algebras.**

1. Let us examine the action of normalising partial isometries on the maximal ideal space (spectrum) of the masa in an arbitrary factor of the limit algebra with a simple example. Let us consider $N_{D_7}(T_7)$. Consider an arbitrary normalising partial isometry v of D_7 in T_7; as mentioned in the remark v has entries either 0 or of absolute value 1 such that each row or column has at most one non-zero entry. The maximal ideal space X of D_7 is the set of its minimal diagonal projections. So $X = \{e_{11}, e_{22}, e_{33}, e_{44}, e_{55}, e_{66}, e_{77}\} \simeq \{1, 2, 3, 4, 5, 6, 7\}$.

$$\text{Let } v = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is easy to check that $vD_7v^* \subseteq D_7$ and $v^*D_7v \subseteq D_7$. Next let us study the action of vxv^* for all $x \in X$. We will observe that this action induces a partial map $\alpha_v : S \to T$ where $S \subseteq X$ and $T \subseteq X$ such that $\alpha_v(f)(d) = f(vdv^*)$; where $f \in X$, $\alpha_v(f) \in X$ and $d \in D$. Now,

- $ve_{11}v^* = 0$,
- $ve_{22}v^* = 0$,
- $ve_{33}v^* = e_{11}$,
- $ve_{44}v^* = 0$,
- $ve_{55}v^* = e_{44}$,
- $ve_{66}v^* = 0$,
- $ve_{77}v^* = e_{22}$.

Let $S = \{e_{33}, e_{55}, e_{77}\} \simeq \{3, 5, 7\}$ and $T = \{e_{11}, e_{44}, e_{22}\} \simeq \{1, 4, 2\}$. Then for $f \in X$ such that $\alpha_v(f) \in X$ there is a partial map $\alpha_v : S \to T$ where $S \subseteq X$ and $T \subseteq X$ such that $\alpha_v(f)(d) = f(vdv^*)$, $d \in D$ is given by

- $\alpha_v(f)(e_{33}) = f(e_{11})$,
- $\alpha_v(f)(e_{55}) = f(e_{44})$,
- $\alpha_v(f)(e_{77}) = f(e_{22})$.

Since $S \subseteq X$ and $T \subseteq X$ are clopen subsets of X, the partial map $\alpha_v : S \to T$ is a partial homeomorphism. Thus here the normalising partial isometry v induces a partial homeomorphism on X, the maximal ideal space of D_7 and this partial homeomorphism essentially moves around the elements of the maximal ideal space of D_7. We do this for all normalising partial isometries and consider all the partial homeomorphisms induced by them. Evidently the spectrum of T_7 coincides with the graphs of all these normalising partial isometries. The topology on X is generated by taking each graph as an open

subset of the spectrum. For the above example since we are in the discrete case, the topology is trivial.

Remark 3.3.1. Next we identify the spectrum or the topological fundamental relation for the refinement limit algebra, the standard limit algebra and the alternating limit algebra. We will work the identification of the spectrum of the refinement limit algebra and the other two will then follow easily. This is mentioned in [16].

2. Let $A = \lim_k (T_{n_k}, \rho_k)$ denote the r^∞ refinement limit algebra, for some positive integer r. Let $B = \lim_k (M_{r^k}, \rho_k)$. Then as seen before $\rho_k(e^k_{ij}) = \sum_{t=1}^{r} e^{k+1}_{(i-1)r^k+t,(j-1)r^k+t}$ where $\{e^k_{ij} : 1 \le i, j \le r^k\}$ is a m.u.s for M_{r^k}. Next we see how to index the matrix units using multi-indices. Let $[r] = \{1, 2, \cdots, r\}$. Then if $k = 2$, $\underline{i} = (i_1, i_2)$ and $\underline{j} = (j_1, j_2)$ are 2-tuples in $[r]^2 = [r] \times [r] = \{1, 2, \cdots, r\} \times \{1, 2, \cdots, r\}$. Thus $\{e_{\underline{i},\underline{j}} : \underline{i}, \underline{j} \in [r]^2\}$ is a m.u.s for M_{r^2}. So in general $\{e_{\underline{i},\underline{j}} : \underline{i}, \underline{j} \in [r]^k\}$ is a m.u.s for M_{r^k}. Let $D = \lim_k (D_{r^k}, \rho_k)$ be a masa which has the m.u.s $\{e_{\underline{i},\underline{i}} : \underline{i} \in [r]^k, k = 1, 2, \cdots\}$. Let X denote the maximal ideal space of D. Given $x \in X$, there is a unique sequence $(e_{\underline{i_1},\underline{i_1}}, e_{\underline{i_2},\underline{i_2}}, e_{\underline{i_3},\underline{i_3}}, \cdots)$ such that $\hat{e}_{\underline{i_n},\underline{i_n}}(x) = 1$ for all n. Conversely each such sequence $e_{\underline{i_1},\underline{i_1}} > e_{\underline{i_2},\underline{i_2}} > e_{\underline{i_3},\underline{i_3}} > \cdots$ corresponds to a unique $x \in X$. Thus it is clear that each decreasing sequence of minimal projections $q^k_x \in D_{r^k}$ corresponds to a unique point $x \in [r]^\infty$ under the correspondence $q^k_x = e_{(x_1,x_2,\cdots,x_k),(x_1,x_2,\cdots,x_k)}$. In this way we identify X, with the Cantor space $[r]^\infty = [r] \times [r] \cdots$ with the product topology. Next we identify the spectrum $R(B)$. To do this we first specify a relationship between successive m.u.s and the natural choice is given by $e_{\underline{i},\underline{j}} = \sum_{m=1}^{r_k} e_{(i_1,\cdots,i_k,m),(j_1,\cdots,j_k,m)}$. Consequently if $E_{\underline{i},\underline{i}}$ denotes the graph of a partial homeomorphism of the maximal ideal space of D_{r^k} induced by $e_{\underline{i},\underline{i}}$ and $E_{\underline{i},\underline{j}}$ denotes the same induced by $e_{\underline{i},\underline{j}}$ then $E_{\underline{i},\underline{i}} = \{(u, u) : u \in \underline{i} \times [r_{k+1}] \times [r_{k+2}] \cdots\}$ and $E_{\underline{i},\underline{j}} = \{(u, v) : u = \underline{i} \times w, v = \underline{j} \times w;\ w \in [r_{k+1}] \times [r_{k+2}] \cdots\}$. Thus we identify the spectrum $R(B)$ with the topological equivalence relation on the Cantor space $X = [r]^\infty$ which consists of all pairs (u, v) of points whose tails coincide eventually. As mentioned before we can visualise the topological relation R for $e_{\underline{i},\underline{j}}$ as *lying over* the square $[0, 1] \times [0, 1]$ by considering the map $\pi(x) = \sum_{k=1}^{\infty} (x_k - 1)r^{-k}$. For details the reader is referred to [16]. In exactly the same way if $n_k = r_1 r_2 r_3 \cdots r_k$ then we may identify $R\{\lim_\to (M_{n_k}, \rho_k)\}$ with an analogous equivalence relation on the Cantor space $[r_1] \times [r_2] \cdots$ We observe that the lexicographic ordering given by $\underline{i} \le \underline{j} \Leftrightarrow \underline{i} = \underline{j}$, or $i_m = j_m$, for $1 \le m < n$, $i_n < j_n$, determines T_{n_k} in such a way that the embedding coincides with the refinement embedding. So $R\{\lim_\to (T_{n_k})\}$= $R\{e_{\underline{i},\underline{j}} : \underline{i} \le \underline{j}, k = 1, 2, \cdots\}$ =$\{(u, v) : u = \underline{i} \times w, v = \underline{j} \times w, \underline{i} \le \underline{j};\ w \in [r_{k+1}] \times [r_{k+2}] \cdots\}$.
3. It is easy to describe the spectrum of the standard limit algebra on the basis of the previous example. We look at $B = \lim_k (M_{n_k}, \sigma_k)$from the previous example. Let A= $\lim_k (T_{n_k}, \sigma_k)$ denote the r^∞ standard limit algebra. Then the spectrum $R(A)$ can be thought of as a subset of $R(B)$ which is determined by the reverse lexicographic order in a manner analogous to the previous example. Again if $n_k = r_1 r_2 r_3 \cdots r_k$ then we may identify $R(B)$ with an

analogous equivalence relation on the Cantor space X as described in the previous example and then obtain $R(A)$ as a subset of $R(B)$ using the reverse lexicographic ordering. Observe that if $x = (x_1, x_2, \cdots)$ and $y = (y_1, y_2, \cdots)$ are points in the Cantor space $X = [r_1] \times [r_2] \cdots$ then $x \leq y$ in the reverse lexicographic order implies that either $x = y$ or $x_k < y_k$ where x_k is the rightmost coordinate of x which differs from y_k.

4. The spectrum for the alternating limit algebra is obtained by alternating the procedures for the standard limit algebra and the refinement limit algebra in the previous examples. For details of this refer to [16].

Remark 3.3.2. The topological equivalence relation or the spectrum is independent of the m.u.s. Stephen Power [15] has used this fact to prove that the spectrum of an AF C*-algebra is a complete isomorphism invariant up to isometric isomorphism. But Donsig, Katsoulis and Hudson [5] have shown that isometric isomorphism is equal to algebraic isomorphism. For TAF algebras, since there is only one canonical masa, the spectrum is an invariant of the algebra. But for a general limit algebra (a limit of digraph algebras), the definition of spectrum depends on the choice of a canonical masa. Since it is not known in general if any 2 canonical masas are inner conjugate, the spectrum may not be independent of the choice of masa. But if the spectra of two AF C*-algebras are isomorphic as topological relations then the algebras themselves are isometrically isomorphic.

4. Ideals in TAF, TUHF and regular canonical subalgebras of AF C*-algebras

4.1. Introduction. The study of various types of ideals of TAF algebras has been a topic of recent research. The structure of various types of ideals has been studied. Among them are the meet-irreducible ideals in [4] and [11], join-irreducible ideals in [8], prime ideals in [9], lie-ideals in [18], n-primitive ideals in [10], Jacobson radical in [3]. This paper studies the principal ideals and gives a large class of limit algebras in which all ideals are principal.

4.2. Principal ideals in Digraph algebras. We will start the study by defining a principal ideal.

Definition 4.2.1. An ideal in a TAF algebra is called principal if it is generated by a single element of the algebra.

We will start by looking at the algebra of upper triangular matrices, T_n. The ideal structure in this finite dimensional algebra is itself complicated and since we are looking at infinite dimensional analogues of these algebras; we will analyze the structure. For example, a generic example of an ideal in T_7 is given by

$$\begin{bmatrix} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

The above ideal is a subset of matrices that vanish at all entries (i, j), for some fixed set in the 7×7 index set of T_7. The boundary of the zero set is described

by a certain non-decreasing function on the diagonal set $\{1, 2, 3, \cdots, 7\}$. In general each ideal I of T_n is described by an order homomorphism $\alpha : \{0, 1, \cdots, n\} \to \{0, 1, \cdots, n\}$ such that $\alpha(k) \leq k$. So $I = I[\alpha] = \{(x_{ij}) : x_{ij} = 0 \text{ whenever } i > \alpha(j)\}$. This is the space of matrices which vanish below the boundary determined by α. Next we claim that all ideals of T_n are principal. This is a known result, but we will include the proof for completeness. Before we write a formal proof we will make some observations that will guide us towards the result. Let $A = (a_{ij})$ be an element of T_n; the algebra of upper triangular matrices. Let $\{e_{ij}\}$ denote the matrix unit system in T_n. Let a_{ij} denote a non-zero entry of A. Then $e_{ii}Ae_{jj}$ yields the matrix with all entries 0 excepting the entry a_{ij}. Then $e_{ij}(k_1e_{j,j} + k_2e_{j,j+1} + k_3e_{j,j+2} + k_4e_{j,j+3} \cdots + k_le_{j,n})$; $(k_1, k_2, k_3, \cdots, k_l$ denotes constants) gives a_{ij} and all the entries in the row containing a_{ij} that are to the right of a_{ij}. Also $(s_1e_{i,i} + s_2e_{i-1,i} + s_3e_{i-2,i} + s_4e_{i-3,i} \cdots + s_me_{1,i})e_{ij}$; $(s_1, s_2, s_3, \cdots, s_m$ denotes constants) gives a_{ij} and all the entries in the column containing a_{ij} that are vertically above a_{ij}. So e_{ij} is a generator for all the entries in the row containing a_{ij} that are to the right of a_{ij} and all the entries in the column containing a_{ij} that are vertically above a_{ij}. In other words, e_{ij} lies in the corner of an $L - block$ matrix and is a generator for that matrix. The ideals of T_n can be visualized as a combination of $L - block$ matrices and so it is apparent what the generator is; it is the sum of elements at the corners of the $L - block$ matrices. Thus every ideal of the algebra is generated by a single element of the algebra and so every ideal is a principal ideal. Thus T_n is a principal ideal algebra. For a formal proof we adopt an approach which gives us foresight in tackling the proof in the infinite dimensional analogue. In fact we will prove that digraph algebras are principal ideal algebras. Since the algebra of upper triangular matrices is a subclass of digraph algebras the proof will also work in that setting.

Theorem 4.2.2. *Let A denote a digraph algebra. If I is an ideal in A then I is a principal ideal.*

Proof: Let $X = \{1, 2, \cdots, n\} \times \{1, 2, \cdots, n\}$. The digraph algebra A is a subalgebra of M_n such that A contains the diagonal matrices. Now given the digraph algebra A, we can find a subset P of $X \times X$ such that $P \subseteq X$ and $A = \{a \in M_n : a_{ij} = 0 \Leftrightarrow (i, j) \notin P\}$. We call P the support set of A. Consider a subset F of P such that $I = \{a \in A : a_{ij} = 0 \Leftrightarrow (i, j) \notin F\}$ is an ideal in A. We make the following observations:

- $(i, i) \in P, \forall i$
- $(i, j) \in P$, $(j, k) \in P \Rightarrow (i, k) \in P$
- $(i, j) \in F$, $(j, k) \in P \Rightarrow (i, k) \in F$
- $(i, j) \in P$, $(j, k) \in F \Rightarrow (i, k) \in F$

To prove that I is a principal ideal we have to obtain a single generator for I. We claim that $g = \sum_{(i,j)\in F} e_{ij}$ is a generator of I where e_{ij} denotes a matrix unit with $(i, j)th$ entry 1 and remaining entries 0. Let I_g denote the ideal generated by g. Now if $e_{ij} \in I$ then $e_{ij} = e_{ii}ge_{jj}$; hence $e_{ij} \in I_g$. Next consider the sum $\sum_{(i,j)\in F} \alpha_{ij}e_{ij}$. Evidently $\sum_{(i,j)\in F} \alpha_{ij}e_{ij} \in I_g$; whence $I_g \subseteq I$. Also in the sum each summand is an element of I. This implies that $I \subseteq I_g$. Thus we have $I = I_g$ and so g is a generator of the ideal I. This implies that I is a principal ideal and so A is a principal ideal algebra.

Remark 4.2.3. The ideal of T_7 shown below can be thought of as a combination of 4 $L-block$ matrices with corners at the (1,1)th, (2,2)th, (6,6)th and (7,7)th entries respectively

$$\begin{bmatrix} * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}.$$

Thus a generator for the above ideal is;

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.3. **Principal ideals in limit algebras.** Since the digraph algebra is a principal ideal algebra, it is natural to think about ideals in their infinite-dimensional analogues, the regular canonical subalgebras. Recall that these limit algebras have the digraph algebras as their building blocks. It is not at all apparent at present if all ideals are principal, although it seems feasible from the previous result. Next we will study the structure of ideals in TAF algebras. It is apparent that UHF algebras have no nontrivial ideals since they are all simple. The ideal structure of AF algebras was analyzed in [1]. The following results about ideals of AF algebras are important.

Definition 4.3.1. A closed subspace S of an approximately finite C*-algebra B is said to be inductive relative to the chain $\{B_k\}$ of B if S is the closed union of the spaces $S \cap B_k$, for $k = 1, 2, \cdots$; i.e. $S = \overline{\cup_k (S \cap B_k)}$

It is very important to note that all ideals of an approximately finite C*-algebra are inductive. The inductivity of ideals plays a very important role in their analysis because this lets us study ideals I in an AF-algebra B by looking at the finite dimensional pieces $I \cap B_k$ of I. An elegant proof for the inductivity of ideals is given in [16]. The essence of the proof is that the injective *-homomorphism $B_k/I \to B_k/(I \cap B_k)$ is an isometry. So if any sequence (b_k) with each element $b_k \in B_k$ converges to an element b in I, then the isometry forces b to lie in $\overline{\cup_k (I \cap B_k)}$. We will state this result as a lemma.

Lemma 4.3.2. *Every closed ideal of an approximately finite C*-algebra B is inductive relative to the subalgebra chain $\{B_k\}$.*

TAF algebras and TUHF algebras have a rich ideal structure. We have already analyzed the structure of ideals in the finite dimensional factors (upper triangular

matrices) of TUHF algebras. TUHF nest algebras have a particularly nice structure. This is also indicated in the next lemma which holds for TUHF nest algebras but does not hold in general. Although the next result is not used in the main result it exposes the restrictions in dealing with a proof for the main result and the subsequent approach. The result is stated as a lemma.

Lemma 4.3.3. *If I_n is an ideal in some finite dimensional factor A_n of a TUHF nest algebra A, for each n, and if I is the ideal in A generated by the I_n, then $I \cap A_n$ is exactly I_n.*

This tells that the process of generating ideals in TUHF nest algebras does not *add* additional elements in the factors. But the above result is not true in general. For the proof of the result and an example illuminating that the process of generating ideals in algebras other than TUHF nest algebras *adds* additional elements in the factors, the reader is referred to [8]. This suggests that generators of an ideal I_n in one of the finite dimensional factors A_n of a limit algebra A may not yield generators of the ideal I generated by I_n in the limit algebra A.

5. Spectrum of ideals in limit algebras

Next we characterize the spectrum for the ideals of TAF algebras and TUHF algebras. Let B be an AF C*-algebra and let I be a closed two-sided ideal in B. By lemma 3.2, I is inductive relative to the subalgebra chain $\{B_k\}$ of B. Let $\{e^k_{ij}\}$ be a matrix unit system for B. Next we define the spectrum of I to be the subset $R(I)$ of $R(B)$ given by $R(I) = \cup\{E^k_{ij} : e^k_{ij} \in I_k, k = 1, 2, \cdots\}$ with the relative topology. Here recall that E^k_{ij} denotes the set of points (x, y) in $X \times X$ of the form $(\alpha(y), y)$ where α is the partial homeomorphism of X induced by e^k_{ij} and y belongs to the domain of α. The topological binary relation $R(I)$ is the spectrum of I. To characterize the spectrum of I we will use the next lemma which is a version of the local spectral theorem for bimodules adapted to ideals. The reader is referred to [16] for a proof of the local spectral theorem for bimodules.

Lemma 5.0.4. *Let e^l_{mn} be an element of the matrix unit system $\{e^k_{ij}\}$ associated with the AF C*-algebra B. If I is an ideal in B and $E^l_{mn} \subseteq R(I)$, then $e^l_{mn} \in I$.*

The next lemma is immediate from the inductivity of ideals and the local spectral theorem. Again we adapt a version of the spectral theorem for bimodules to ideals. The Bimodule spectral theorem is proved in [16].

Lemma 5.0.5. *Let I and J be ideals in the AF C*-algebra B. If $R(I) = R(J)$ then $I = J$.*

6. Groupoid terminology for the spectrum

6.1. **Introduction.** The spectrum $R(B)$ of an AF C*-algebra B is an example of an approximately finite r-discrete principal groupoid. In this section we will discuss what this means. AF C*-algebras are groupoid C*-algebras and it is the structure and substructures of the groupoid which helps in the coordinization of these algebras. Any closed subalgebra of an AF C*-algebra also has a functional representation in terms of an open subset of the spectrum and we will make this observation in this section.

6.2. **Principal Groupoids.** To begin with let us define a groupoid.

Definition 6.2.1. A groupoid is a set G together with a subset $G^2 \subset G \times G$, a product map $(a, b) \to ab$ from G^2 to G, and an inverse map $a \to a^{-1}($ so that $(a^{-1})^{-1} = a$) from G to G such that:

- If $(a, b), (b, c) \in G^2$, then $(ab, c), (a, bc) \in G^2$ and $(ab)c = a(bc)$;
- $(b, b^{-1}) \in G^2$ for all $b \in G$, and if $(a, b) \in G^2$, then $a^{-1}(ab) = b$, $(ab)b^{-1} = a$.

A trivial example of a groupoid is a group. Another example of a groupoid is an equivalence relation R on a set X. In this case $R^2 = \{((x, y), (y, z)) : (x, y), (y, z) \in R\}$. R is a groupoid with the product map from R^2 to R given by $((x, y), (y, z)) \to (x, z)$ and the inverse defined as $(x, y)^{-1} = (y, x)$. Groupoids based on equivalence relation are called *principal groupoids*.

Definition 6.2.2. The unit space G^0 of a groupoid G is defined to be the set $\{xx^{-1} : x \in G\}$.

Definition 6.2.3. The range map is the map $r : G \to G^0$ given by $r(x) = xx^{-1}$ and the source map is the map $d : G \to G^0$ given by $d(x) = x^{-1}x$.

For principal groupoids the range and the source maps are given by

- $r(x, y) = (x, y)(x, y)^{-1} = (x, y)(y, x) = (x, x)$
- $d(x, y) = (x, y)^{-1}(x, y) = (y, x)(x, y) = (y, y)$

We say that G is a topological groupoid if G is equipped with a suitable topology for which product and inversion are continuous . When G is a topological groupoid, we also require that the range and the source maps are partial homeomorphisms.

Lemma 6.2.4. *Let G be a locally compact groupoid. Then each of r, d is an open map from G onto G_0.*

For a proof of the above lemma refer to [13]. For any locally compact groupoid G, let G^{op} denote the family of open subsets A of G such that the restrictions r_A, d_A of r, d to A are homeomorphisms onto open subsets of G.

Definition 6.2.5. An $r-discrete$ groupoid is a locally compact groupoid G such that G^{op} is a basis for the topology of G.

Definition 6.2.6. G-sets are subsets of a topological groupoid G such that the restrictions of the range and domain functions are one-to-one.

Note that every $A \in G^{op}$ is a G-set.

Definition 6.2.7. Let P be an open subset of G containing G^0. P is called a partial order in G if $P \circ P \subseteq P$ and $P \cap P^{-1} = G^0$. Moreover, if $P \cup P^{-1} = G$ then P is called a total order in G. If $P \circ P \subseteq P$ and $P = P^{-1}$, then we call P an equivalence relation on a subgroupoid of G

Lemma 6.2.8. *If P is a total order on G then P is closed.*

Remark 6.2.9. A TAF algebra is strongly maximal if and only if P is totally ordered. For proofs refer to [12]. Also observe that in this case P is clopen.

6.3. **Groupoid structure of AF C*-algebras.** Next we will observe that an AF C*-algebra B can be expressed as $C^*(G)$ for a suitable principal groupoid G; i.e $B \approx C^*(G)$. We begin by reviewing the construction of a groupoid *-algebra, $C^*(G)$, from a locally compact, r-discrete, principal groupoid G. For the construction in a more general setting, see [12]. Let $C_c(G)$ denote the family of continuous complex-valued functions with compact support. We can make $C_c(G)$ into a topological *-algebra by defining for $f, g \in C_c(G)$, $(f * g)$ and f^* by,

- $(f * g)(a, b) = \sum_{((a,c),(c,b)) \in G^2} f(a, c)g(c, b)$.
- $f^*(a, b) = \overline{f(b, a)}$.

With these operations $C_c(G)$ is a $* - algebra$. Let $C^*(G)$ denote the completion of $C_c(G)$ in a natural norm (as defined in Muhly and Solel's paper [12]). Since we are assuming that the groupoid G is r-discrete and principal, $C^*(G)$ can be viewed as a subspace of continuous functions on G.([17], 4.2) We consider the space of continuous functions with compact support on G^0 i.e. $C_c(G^0)$ and we identify the closure of this space in $C^*(G)$ by $C_0(G^0)$. The next result is a consequence of the Spectral theorem of Bimodules; for a proof the reader is referred to [12]. For the sake of completeness we state the result.

Lemma 6.3.1. *Suppose $B \subseteq C^*(G)$ is a closed $C^*(G^0) - bimodule$. Let $Q(B) = \{(x, y) \in G : b(x, y) = 0; \forall b \in B\}$ and $I(Q) = \{b \in C^*(G) : b = 0$ on $Q\}$. Then $B = I(Q(B))$.*

Remark 6.3.2. Again by the Spectral theorem of Bimodules, for any subalgebra A of B such that A contains $C_0(G^0)$(here $C_0(G^0)$ is the analogue of the diagonal matrices in the context of AF C*-algebras), there is a subset P of G such that A consists of all the elements in $C^*(G)$ that are supported on P. Consequently we denote A by $A(P)$. Thus TUHF, TAF and regular canonical algebras have representations in the form $A(P)$.

6.4. **Examples of Groupoid representations of AF C*-algebras.** We will observe the above representations for a few examples. As we have mentioned before, the spectrum $R(B)$ of an AF C*-algebra B is an example of an r-discrete principal groupoid. Also we have noted that the underlying space for the groupoid $R(B)$ is the maximal ideal space X for a canonical masa; and $R(B) \subset X \times X$. Let us recall that given an AF C*-algebra B and a m.u.s, each matrix unit, v, from the m.u.s acts on the diagonal D of B by conjugation ($v^*Dv \subseteq D$). Consequently each matrix unit v induces a partial homeomorphism of X (maximal ideal space of D) onto itself. We denoted the graph of this homeomorphism by $\hat{v}$. Then the graphs of all the partial homeomorphisms induced by matrix units (matrix units suffice) is the spectrum $R(B)$ of B and is a groupoid, say G. Thus $G \subseteq X \times X$ and is an equivalence relation. We put a topology on G and this topology is the smallest topology in which every $\hat{v}$ is an open subset. G as defined above is an example of an r-discrete, principal, topological groupoid. The graph, $\hat{v}$, of the partial homeomorphism associated with a matrix unit(or a normalising partial isometry) has the following properties:

- (x, y_1) and $(x, y_2) \in v \Rightarrow y_1 = y_2$.
- (x_1, y) and $(x_2, y) \in v \Rightarrow x_1 = x_2$.

As mentioned before a subset of G with these properties is called a G-set. From the above discussion we observe that an r-discrete principal groupoid G such that $G \subset X \times X$, where X is a topological space, satisfies the following conditions:

- G is a locally compact Hausdorff space.
- The map $(x, y) \to (y, x)$ from G to G, and the product map from G^2 to G given by $((x, y), (y, z)) \to (x, z)$ are continuous.
- The map $x \to (x, x)$ from X to G is a homeomorphism.
- The unit space G^0 is an open subset of G.

Remark 6.4.1. In the second condition the product map is a partially defined map on G^2 and hence carries the relative product topology of the topological space $X \times X$. The fourth condition is also called $(r - discreteness)$ because it implies that for each $x \in X$ the set $G \cap \{(x, y) : y \in X\}$ is a discrete space in the relative topology.

The spectrum of an AF C*-algebra satisfies all the above conditions. Next we look at the 2^∞ refinement and the standard embedding algebras and describe their spectra as a groupoid G where $G \subset X \times X$ and X is a topological space.

- Let $X = \{(a_n) : a_n \in \{0, 1\}\}$. Set $G = \{(a, b) \in \ X \times X : a_n \neq b_n$ for a finite no: of $n's$ $\}$. The set X is a locally compact, second countable, Hausdorff space and $G \subseteq X \times X$ is a second countable, locally compact, r-discrete principal groupoid. We will identify G with the spectrum of the refinement limit algebra A. The groupoid operations on G are as follows: If $x = (a, b)$ and $y = (c, d)$ are in G then $xy = (a, d)$ if $b = c$ and it is undefined otherwise. As seen above the range and domain functions for G are defined by $r((x, y)) = (x, x)$ and $d((x, y)) = (y, y)$. In the present situation there are G-sets of the form

$$E_{(a_1,a_2,a_3,\cdots,a_n),(b_1,b_2,b_3,\cdots,b_n)} = \{(a_1, a_2, a_3, \cdots, a_n, x_{n+1}, x_{n+2}, x_{n+3}, \cdots), \\ (b_1, b_2, b_3, \cdots, b_n, x_{n+1}, x_{n+2}, x_{n+3}, \cdots) : x_k \in \{0, 1\} \forall k \geq n + 1\}$$

  These sets are compact and open and forms a base for the topology of G. G is r-discrete because the unit space $\{(x, x) : x \in X\} = G^0$ is open. Also $G^2 \subseteq$ G and is the set $\{((a, b), (b, c)) : a, b, c \in X\}$. Let $C_c(G)$ denote the space of all continuous complex-valued functions with compact support on G. Recall that for $f, g \in C_c(G)$, $f * g$ and f^* on G^2 are given by
  1. $(f * g)(a, b) = \sum_{((a,c),(c,b)) \in G^2} f(a, c)g(c, b)$
  2. $f^*(a, b) = \overline{f(b, a)}$.

  With these operations $C_c(G)$ is a $* - algebra$. $C^*(G)$ denotes the completion of $C_c(G)$ in a natural norm as observed before. Also $C^*(G)$ can be viewed as a subspace of continuous functions on G. For any G-set E, χ_E, which denotes the characteristic function of E, is a partial isometry in $C^*(G)$. Consequently any element of $C^*(G)$ can be written as a norm limit of linear combinations of $f * \chi_E$, where E is a G-set and $f \in C_c(G^0)$. Also if E and F are G-sets, then $\chi_E * \chi_F = \chi_{EF}$. Now, given the refinement limit algebra A we can find an open subset P of G containing G^0 such that A has a representation as functions supported on P. Let $P = \{(a, b) : (a_1, a_2, ...a_N)$ precedes $(b_1, b_2, ...b_N)$ in lexicographic order and $a_n = b_n$ for $n > N\}$. Then P is an open subset of G containing G^0. P is a total order and uniquely determines A. We observe that P is the spectrum of A. A is a subalgebra of $C^*(G)$

which satisfies the condition that its meet with its adjoint is $C_c(G^0)$ and $A = \{f \in C^*(G) : f(h) = 0, \forall h \in G\backslash P\}$. Thus by the Spectral theorem of Bimodules we have that $A \approx A(P)$.

- By imitating the above argument with $P = \{(a, b) : (a_1, a_2, ...a_N)$ precedes $(b_1, b_2, ...b_N)$ in reverse lexicographic order and $a_n = b_n$ for $n > N\}$ we obtain a representation of the standard limit algebra A as $A(P)$.

Remark 6.4.2. The above procedure can be generalized to TAF, TUHF and regular canonical algebras. Let A denote a TAF, TUHF or a regular canonical algebra. Since we are dealing with ideals we note that if I is an ideal of A then we can find an open subset F of P such that I has a representation as functions supported on F. F is called the ideal set of I.

7. Principal ideals in regular canonical subalgebras of AF C*-algebras

7.1. **Introduction.** We have already seen that a TAF algebra A can be represented in the form $A(P)$ for an open set P of the groupoid for the enveloping $C^* - algebra$. Moreover $A(P)$ is strongly maximal if and only if P is totally ordered and in this case each factor A_n of A can be represented as a direct sum of upper triangular matrices, each upper triangular matrix obtained from its corresponding full matrix algebra. Also for strongly maximal TAF algebras the embeddings $j_n : A_n \rightarrow A_{n+1}$ are *-extendible to their appropriate full matrix algebras. We will first prove that ideals in subalgebras of second countable, locally compact, r-discrete principal groupoids are principal ideals. This will be the main result in this paper. Since the spectrum of regular canonical subalgebras is a locally compact, second countable, r-discrete, principal groupoid it will follow that the regular canonical subalgebra is a principal ideal algebra and so are strongly maximal TAF algebras and strongly maximal in factors TUHF algebras. We will obtain these results as corollaries to the main result. It is important to mention that the proofs to the corollaries will use regularity of embeddings and this is the main characteristic of the above limit algebras.

7.2. **Principal ideals in subalgebras of certain groupoid C*-algebras.** Before we start with the main result, let us recall that the spectral theorem for Bimodules (lemma 6.3.1) implies that there is a one-to-one correspondence between ideals I of a subalgebra A of a groupoid C*-algebra G and open subsets F of P such that P∘F∘P ⊆F. This correspondence is given by $I = \{f \in C^*(G) : f(h) = 0, \forall h \in P\backslash F\}$. F is called the ideal set of I. The next lemma is also relevant. For a proof the reader is referred to [12].

Lemma 7.2.1. *For each partial order P in G, $A(P)$ is a norm closed subalgebra of $C^*(G)$ containing $C_0(G^0)$. Conversely, each subalgebra A of $C^*(G)$ containing $C_0(G^0)$ is of the form $A(P)$ for a unique partial order P. The correspondence $P \mapsto A(P)$ is an inclusion preserving bijection between the collection of partial orders in G and norm closed subalgebras of $C^*(G)$ containing $C_0(G^0)$.*

Next we state and prove the main result in this paper.

Theorem 7.2.2. *Let G denote a second countable, locally compact, r-discrete principal groupoid that admits a cover by compact open G-sets. Let A denote a subalgebra of $C^*(G)$ such that $C_0(G^0) \subseteq A$. Then A is a Principal Ideal Algebra.*

Proof: Since G is second countable, G has a countable basis of open sets. Thus every compact open G-set is the union of a finite number of these open sets. Consequently if G has a basis of compact open G sets then it has a countable basis. Now, given A, let P denote the open subset of G containing G^0 such that $A = A(P)$. Since G is covered by countably many compact open G-sets and A is a subalgebra of G such that $C_0(G^0) \subseteq A$, P is covered by the G-sets it contains. As mentioned above there is a one-to-one correspondence between ideals I of A and open subsets F of P such that $P \circ F \circ P \subseteq F$. This correspondence is given by $I = \{f \in C^*(G) : f(h) = 0, \forall h \in P\backslash F\}$ where F denotes the ideal set of I. Next there exist countably many compact open G-sets K_i such that $\cup_{i=1}^{\infty}\{K_i\} = F$. We will write $\cup_{i=1}^{\infty}\{K_i\} = F$ as a countable disjoint union. Let

- $E_1 = K_1$,
- $E_2 = K_2\backslash K_1 = K_2 \cap {K_1}^c$,
- $E_3 = K_3\backslash(K_1 \cup K_2) = K_3 \cap (K_1 \cup K_2)^c$,

  $\vdots$
- $E_i = K_i\backslash(K_1 \cup K_2 \cdots \cup K_{i-1}) = K_i \cap (K_1 \cup K_2 \cdots \cup K_{i-1})^c$,

  $\vdots$

These E_i are countable, disjoint, compact and open. Also $\cup_{i=1}^{\infty}\{E_i\} = F$. For each E_i, χ_{E_i} denotes the characteristic function of $E_i \subseteq G$. Since E_i is compact and open, $\chi_{E_i} \in C_c(G) \subseteq C^*(G)$. Next consider the sequence (χ_{E_i}). Now since $\cup_{i=1}^{\infty}\{E_i\} = F$ we claim that the sequence (χ_{E_i}) generates I. To prove this we first observe that any arbitrary deleted G-set K_i (or made smaller by the deletion process) can be obtained as follows. Now, $E_1 = K_1$ and so $\chi_{K_1} = \chi_{E_1}$. Next, $E_2 = K_2\backslash K_1$ and so $K_2 = E_2 \cup (K_1 \cap K_2)$. But $E_1 = K_1$ and so $K_2 = E_2 \cup (E_1 \cap K_2)$. This implies $\chi_{K_2} = \chi_{E_2} + \chi_{E_1}\chi_{K_2}$. Note that $\chi_{E_1} \in I$ and so $\chi_{E_1}\chi_{K_2} \in I$. Consequently $\chi_{E_2} + \chi_{E_1}\chi_{K_2} \in I$ and so $\chi_{K_2} \in I$. Next, $E_3 = K_3\backslash(K_1 \cup K_2)$ and so $K_3 = E_3 \cup [K_3 \cap (K_1 \cup K_2)]$. But $E_1 \cup E_2 = K_1 \cup K_2$ and so $\chi_{K_3} = \chi_{E_3} + \chi_{K_3}(\chi_{E_1} + \chi_{E_2})$. Again by definition of an ideal, $\chi_{K_3} \in I$. In general for any integer i, $E_i = K_i\backslash(K_1 \cup K_2 \cdots \cup K_{i-1})$ and so $K_i = E_i \cup [K_i \cap (K_1 \cup K_2 \cup K_3 \cdots \cup K_{i-1})]$. But $E_1 \cup E_2 \cdots \cup E_i = K_1 \cup K_2 \cdots \cup K_i$ and so $\chi_{K_i} = \chi_{E_i} + \chi_{K_i}(\chi_{E_1} + \chi_{E_2} + \chi_{E_3} + \cdots + \chi_{E_{i-1}})$. Consequently by definition of an ideal, $\chi_{K_i} \in I$. Thus $\chi_{K_i} \in I$, for all i. Therefore, the sequence (χ_{E_i}) generates I. Next we claim that $\sum_{i=1}^{\infty} \frac{\chi_{E_i}}{2^i}$ is a generator of the ideal I. Since the partial sums $\sum_{i=1}^{n} \frac{\chi_{E_i}}{2^i}$, $n = 1, 2, \cdots$ are norm convergent, let $g = \sum_{i=1}^{\infty} \frac{\chi_{E_i}}{2^i}$. The convergence of partial sums $\sum_{i=1}^{n} \frac{\chi_{E_i}}{2^i}$, $n = 1, 2, \cdots$ assures that $g \in I$. Let I_g denote the ideal generated by g and E_g denote the ideal set of I_g. We will prove that $I_g = I$. It is evident that $I_g \subseteq I$. In other words, ideal set of $I_g \subseteq$ ideal set of I i.e. $E_g \subseteq F$. To prove $I \subseteq I_g$, it would suffice to prove that $\chi_{E_j} \in I$ implies that $\chi_{E_j} \in I_g$, for any arbitrary j. Now we have that $\cup_{i=1}^{\infty}\{E_i\} = F$. Next consider $\chi_{E_j} \in I$, for some j. Let $d(\chi_{E_j})$ and $r(\chi_{E_j})$ denote the domain and range projections of χ_{E_j}, we claim that $r(\chi_{E_j})gd(\chi_{E_j}) = \frac{(\chi_{E_j})}{2^j}$.

Now, $r(\chi_{E_j})gd(\chi_{E_j})$
$= r(\chi_{E_j})(\sum_{i=1}^{\infty} \frac{\chi_{E_i}}{2^i})d(\chi_{E_j})$
$= \sum_{i=1}^{\infty} r(\chi_{E_j})\frac{\chi_{E_i}}{2^i}d(\chi_{E_j})$
$= \frac{\chi_{E_j}}{2^j}$.

But $g \in I_g$, so this shows $\frac{\chi_{E_j}}{2^j} \in I_g$, and hence $\chi_{E_j} \in I_g$. Thus $E_j \subseteq E_g$, for all j. In other words, the ideal set of $I \subseteq$ ideal set of I_g and so $I_g = I$.

Thus if I is an ideal of A then it is a principal ideal and the subalgebra A of G is a principal ideal algebra.

7.3. **Principal ideals in regular canonical subalgebras of AF C*-algebras.** We will need to define *subordinate* of a matrix unit before we get into the corollary. Let A be a regular canonical subalgebra of an AF C*-algebra B and let us denote the presentation of A by $A_1 \overset{\varphi_1}{\to} A_2 \overset{\varphi_2}{\to} A_3 \overset{\varphi_3}{\to} A_4 \overset{\varphi_4}{\to} \cdots A_n \overset{\varphi_n}{\to} A_{n+1} \cdots$ with star injections $\varphi_n : A_n \to A_{n+1}$. Also for $m > n$, define $\varphi_{m,n} :\ A_n \to A_m$ to be the embedding from A_n to A_m; i.e. $\varphi_{m,n} = \varphi_{m-1} \circ \cdots \circ \varphi_n$.

Definition 7.3.1. If $v \in A_n$ is a partial isometry and $m \geq n$, a partial isometry $u \in A_m$ is a subordinate of $\varphi_{m,n}(v)$ if $r(u)\varphi_{m,n}(v)d(u) = u$.

Corollary 7.3.2. *Let A denote a regular canonical subalgebra of an AF C*-algebra B. Then A is a principal ideal algebra.*

Proof: Let $A = A_1 \overset{\varphi_1}{\to} A_2 \overset{\varphi_2}{\to} A_3 \overset{\varphi_3}{\to} A_4 \overset{\varphi_4}{\to} \cdots A_n \overset{\varphi_n}{\to} A_{n+1} \cdots$ denote an injective direct system (presentation) of C*-algebras $A_n, n = 1, 2, \cdots$ with star-extendible injections $\varphi_n : A_n \to A_{n+1}$. Since the matrix unit system need not be unique we choose a matrix unit system such that every matrix unit in B_n is a sum of matrix units in B_{n+1}. Let D_n denote the diagonal of B_n, for each n and $lim_{n\to\infty} D_n = D$ denote the canonical masa in B. The spectrum of B has the structure of an r-discrete principal groupoid, say G, where $G \subseteq X \times X$ and X is the spectrum of D. Also G is a second countable, locally compact, admits a cover by compact open G-sets. Note that in this case the compact open G-sets are supports of matrix units. Thus G satisfies the hypothesis of the main result. Consequently we can obtain $C_c(G)$, the space of all continuous complex-valued functions with compact support on G as a $* - algebra$ and $C^*(G)$ as the completion of $C_c(G)$ with respect to a suitable norm. Now, given A, let P denote the open subset of G containing G^0 such that $A = A(P)$. Thus G is a second countable, locally compact, r-discrete principal groupoid that admits a cover by compact open G-sets and A is a subalgebra of G such that $C_0(G^0) \subseteq A$. Consequently A is a Principal Ideal Algebra by Theorem 7.2.2.

Remark 7.3.3. The above proof is more concrete than the proof of the theorem in the sense that we have a presentation of the regular canonical subalgebra A of the AF C*-algebra B with supports of matrix units providing the compact open G-sets. In this spirit we describe the process of obtaining a disjoint collection of compact open G-sets E_i such that $\cup_{i=1}^{\infty}\{E_i\} = F$, where F denotes the ideal set of an ideal I in A in the proof of the above corollary. Once this collection is obtained the remaining part of the proof is the same as in Theorem 7.2.2. As mentioned in the corollary P denotes the open subset of G containing G^0 such that $A = A(P)$. We have already chosen the matrix unit system in such a way that every matrix unit in A_n is a sum of matrix units in A_{n+1}. At this stage we look at the intersection of I with individual factors A_n; i.e. $A_n \cap I$, for each n. Now since $\varphi_n(A_n) \subseteq A_{n+1}$ we will have overlapping matrix units in $A_n \cap I$ and $A_{n+1} \cap I$; for each n and consequently intersecting compact open G-sets in the sequence of algebras, $A_n \cap I; n = 1, 2, \cdots$. For each compact open G-set E_i which is a support of a matrix unit, let χ_{E_i} denote

the characteristic function on E_i. Then χ_{E_i} is a matrix unit in the sequence of algebras $A_n \cap I; n = 1, 2, \cdots$. We will obtain a collection of G-sets $\{E_i\}$ from this sequence in such a manner that these G-sets are supports of matrix units in the sequence of algebras and $\cup_{j=1}^{\infty}\{E_i\} = F$. Let us first list the matrix units of $A_n \cap I$; $n = 1, 2, \cdots$ in order starting with matrix units of $A_1 \cap I$. Let χ_{K_1}, χ_{K_2}, χ_{K_3}, $\chi_{K_4}, \cdots$ denote this list or sequence. This sequence consists of all the matrix units of the sequence of algebras $A_n \cap I$; $n = 1, 2, \cdots$ in order. Now to achieve our aim of obtaining the required collection of G-sets E_i in the sequence of algebras $A_n \cap I$; $n = 1, 2, \cdots$ we delete the matrix units in $A_n \cap I$; $n = 2, 3, 4, \cdots$ which are subordinate to a previous matrix unit. After this deletion process let χ_{E_1}, χ_{E_2}, χ_{E_3}, $\chi_{E_4}, \cdots$ be the sequence of the remaining matrix units. In this sequence no matrix unit is a subordinate of a previous matrix unit and consequently these matrix units are nonoverlapping and so their supporting G-sets E_i are nonintersecting. We also have that $\cup_{i=1}^{\infty}\{E_i\} = F$. We claim that the sequence (χ_{E_i}) generates I. To prove this we first observe that any arbitrary deleted G-set K_i can be obtained as follows. Let E_i be a G-set from $\cup_{j=1}^{\infty}\{E_i\}$ such that χ_{K_i} is subordinate to χ_{E_i}. That is to say that if $\mathrm{r}(K_i)$ and $\mathrm{d}(K_i)$ denote the range and source maps on the groupoid then we have $K_i = \mathrm{r}(K_i)E_i\mathrm{d}(K_i)$. Now $\mathrm{r}(K_i)$ and $\mathrm{d}(K_i)$ are subsets of G_0 which is the unit space of G. So from the equation $K_i = \mathrm{r}(K_i)E_i\mathrm{d}(K_i)$ we observe that K_i is in F. Observe that the fact that E_i is in F and that F is an ideal set, automatically places K_i in F. So we conclude that the sequence (χ_{E_i}) generates I. From this point on the remaining portion of proof is as in Theorem 7.2.2.

Remark 7.3.4. We recall that strongly maximal TAF algebras, strongly maximal in factors TUHF algebras and in general, TAF algebras are regular canonical subalgebras and so the theorem holds in those settings.

References

[1] O. Bratteli, *Inductive limits of finite dimensional C*-algebras*, Trans. Amer. Math. Soc. **171** (1972), 195–234.

[2] K. R. Davidson, *C*-algebras by example*, Fields Institute monographs, no. 6, American Math.Soc., Providence, RI (1996).

[3] A. P. Donsig, *Semisimple triangular AF algebras*, J. Funct. Anal. **111** (1993), 23–349.

[4] A. P. Donsig, A. Hopenwasser, T. D. Hudson, M. P. Lamoureux, and B. Solel, *Meet irreducible ideals in direct limit algebras*, Math. Scand., to appear.

[5] A. P. Donsig, T. D. Hudson, and E. G. Katsoulis, *Algebraic isomorphisms of limit algebras*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 1169–1182 (electronic).

[6] J. Glimm, *On a certain class of operator algebras*, Trans. Amer. Math. Soc. **95** (1960), 318–340.

[7] A. Hopenwasser and S. C. Power, *Classification of limits of upper triangular matrix algebras*, Proc. Edinburgh Math. Soc. **36** (1992), 107–121.

[8] T. D. Hudson, *Ideals in triangular AF algebras*, Proc. London Math. Soc. (3) **69** (1994), 345–376.

[9] ———, *Radicals and prime ideals in limit subalgebras of AF algebras*, Quart. J.Math.Oxford Ser. (2) **48** (1997), 213–233.

[10] M. P. Lamoureux, *Nest representations and dynamical systems*, J. Functional Analysis **114** (1993), 467–492.

[11] ———, *The topology of ideals in some triangular AF algebras*, J. Operator Theory **37** (1997), 91–109.

[12] P. S. Muhly and B. Solel, *Subalgebras of groupoid C*-algebras*, J. Reine Angew. Math. **402** (1989), 41–75.

[13] A. L. T. Paterson, *Groupoids, inverse semigroups, and their operator algebras*, Progress in Mathematics **170** (1996).
[14] J. R. Peters, Y.-T. Poon, and B. H. Wagner, *Triangular AF algebras*, J. Operator Theory **23** (1990), 81–114.
[15] S. C. Power, *Classification of tensor products of triangular operator algebras*, Proc. London Math. Soc. **61** (1990), 571–614.
[16] ______, *Limit algebras: An introduction to subalgebras of C^*-algebras*, Pitman Research Notes in Mathematics Series, vol. 278, Longman Scientific and Technical, England, New York, 1992, Errata available at http://www.maths.lancs.ac.uk/∼power/pubs.brief.html.
[17] J. Renault, *A groupoid approach to C^*-algebras*, Lect. notes in math. **793** (1980).
[18] T.D.Hudson, L.W.Marcoux, and A.R.Sourour, *Lie ideals in triangular operator algebras*, Trans. Amer. Math. Soc. **350** (1998), 3321–3339.

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